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AN APPLICATION OF NONLINEAR FUNDAMENTAL PROBLEMS OF A TRANSVERSELY ISOTROPIC LAYER IN FINITE ELASTIC DEFORMATION[1]

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Abstract

We obtain the complex potentials and the accompanying nonlinear fundamental problems for the plane problem of a transversely-isotropic body under finite elastic deformation. The fundamental problem-two is considered for an infinite medium, with a circular hole in the initial configuration. It is obtained that in the current configuration the deforming contour is not rigidly circular, due to finite deformation effect. Variation of the deforming contour with respect to the parameter of finite deformation (or parameter of nonlinearity) is given.

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[1] This is the second part on this topic. But for self-containment it includes the first sections of the other part, "On Application of Complex Variable Method to Plane Problem of a Transversely Isotropic Body in Finite Elasticity", which introduces the topic and considers in detail the 1st Fundamental Problem.

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INTRODUCTION

Using the complex analysis method of investigation in plane elasticity, problems are reduced to what Muskhelishvili [1] called *Fundamental Problems.* Here, we highlight the fundamental problems one and two. Fundamental Problem One *FP I*: find the state of elastic equilibrium of a body for a given applied load on its boundary. Fundamental Problem Two *FP II*: find the state of elastic equilibrium of a body, given displacement on its boundary. For small deformation elasticity these fundamental problems are linear. However, when the elastic deformation in view is finite, these problems are nonlinear; consistent with what is already known of finite elasticity in general.

To obtain a fundamental problem there is the need to first of all establish a pertinent formulation of the complete boundary value problem of elasticity. In finite deformation theory [2,3], the latter centres on establishing an appropriate energy function: when anisotropy is involved, then the difficulty in doing this becomes multiple. This function, amongst other things, must be such that it will ensure energy conjugacy between the geometric characteristic (strain) and the mechanical characteristic (stress).

We consider the equilibrum of an elastic medium Ω, which by Truesdell [4] is a manifold of particles, in a three-dimensional euclidean space E^3. In the initial (or reference) configuration it occupies space $\Omega_o \in E^3$ with the boundary Σ_o, characterized by unit normal vector $\vec{n}$ and loaded by force $\vec{f_o}$. We then sort for the current configuration $\Omega \in E^3$ of this body, now with boundary Σ characterized by normal vector $\vec{N}$ and loaded by force $\vec{f}$. That is, as a result of load $\vec{f^o}$ applied on Σ_o the reference medium has changed to Ω loaded on the boundary Σ now by force $\vec{f}$ (see fig.1). Aside from giving an answer to the question of what tractable energy function provides necessary conjugacy, as mentioned above, there is the problem of knowing also what is $\vec{f}$ on a changing surface Σ. Often, $\vec{f}$ is taken to be a "dead load", e.g. pressure on the boundary.

Here, on the basis of the energy function of John [5], for an isotropic semi-linear material an anisotropic equivalent for a transversely-isotropic body [6] is used . We then constitute the pertinent boundary value problem. A complex variable formulation is done and the consequent nonlinear fundamental problems are obtained. On the basis of this, the problem of infinite region with a hole is considered and the nonlinear effect of finite deformation is investigated.

1 PROBLEM SETTING

Geometry of Deformation

Let Ω be a transversal isotropic medium in three dimensional euclidean space E^3. We look at the equilibrum state of Ω in plane finite deformation.

The deformation of Ω is given by specifying the position vector $\vec{r}$ of a particle prior to deformation in the initial (reference) configuration Ω_o with the boundary Σ_o and its orientation normal unit vector $\vec{n}$ and the position vector $\vec{R}$ in the current configuration Ω with the boundary Σ and its orientation normal vector $\vec{N}$:

$$\vec{r} = a^1\vec{e_1} + a^2\vec{e_2} + a^3\vec{e_3} \tag{1.1}$$

$$\vec{R} = x^1\vec{e_1} + x^2\vec{e_2} + x^3\vec{e_3} \tag{1.2}$$

such that for plane deformation:

$$x^\alpha = x^\alpha(a^1, a^2); \quad x^3 = ka^3; \quad \alpha = 1, 2 \tag{1.3}$$

where, a^m, x^m are the material coordinates in Ω_o and Ω respectively; $\vec{e_m}$ is now defined as the orthonormal basis $\vec{e_1} = \vec{i}, \vec{e_2} = \vec{j}, \vec{e_3} = \vec{k}$; $m = 1, 2, 3$; k is any real constant.

Let the geometry of deformations be the tensor-gradient of the position vector $\vec{R}$ in $\Omega(\vec{R})$ taking in the initial configuration $\Omega_o(\vec{r})$. That is, applying the operator of gradient-vector in the reference configuration, $\overset{o}{\nabla} \equiv \vec{e_m}\frac{\partial}{\partial a^m}$, on the position vector $\vec{R}$ in the current configuration, we obtain the tensor-gradient (or *deformation gradient*):

$$\overset{o}{\nabla}\vec{R} = \vec{e_\alpha}\vec{e_\beta}\frac{\partial x^\beta}{\partial a^\alpha} + \vec{e_3}\vec{e_3}k. \tag{1.4}$$

We also consider the *deformative rotation tensor* of the medium:

$$\tilde{O}^D = \tilde{U}^{-1}\cdot \overset{o}{\nabla}\vec{R} = \tilde{E}\cos\chi + (1-\cos\chi)\vec{e_3}\vec{e_3} - \vec{e_3}\times\tilde{E}\sin\chi \tag{1.5}$$

where, $\tilde{U}$, such that $\tilde{U}^2 = \overset{o}{\nabla}\vec{R}\cdot \overset{o}{\nabla}\vec{R}^T$, is the symmetric stretch tensor, arising from the polar decomposition of the deformation gradient, $\overset{o}{\nabla}\vec{R} = \tilde{U}\cdot\tilde{O}^D$; $\tilde{E}$ is the unit tensor in E^3 and

$$\cos\chi = \frac{1}{q}\left(\frac{\partial x^1}{\partial a^1} + \frac{\partial x^2}{\partial a^2}\right), \sin\chi = \frac{1}{q}\left(\frac{\partial x^2}{\partial a^1} - \frac{\partial x^1}{\partial a^2}\right); q = \sqrt{\left(\frac{\partial x^1}{\partial a^1} + \frac{\partial x^2}{\partial a^2}\right)^2 + \left(\frac{\partial x^2}{\partial a^1} - \frac{\partial x^1}{\partial a^2}\right)^2}. \tag{1.6}$$

For any functions ϕ and ψ, here and elsewhere we denote their dot product and cross product respectively by $\phi\cdot\psi$ and $\phi\times\psi$.

Static Equation for Transversely Isotropic Material

We look at the equilibrium state of Ω in plane finite deformation. For this, we first recall the energy function for an isotropic semi-linear material in finite deformation [3,5]:

$$W = \mu S_2 + 1/2\lambda S_1^2 \tag{1.7}$$

where, S_1 and S_2 are the invariants of the deformation geometry, $S_1 = \tilde{E}\cdot\cdot(\tilde{U}-\tilde{E}) \equiv I_1(\tilde{U}-\tilde{E})$, $S_2 = I_1(\tilde{U}-\tilde{E})^2$. λ and μ are the Lame constants.

On the basis of (1.7) we take the energy for a transversely isotropic semi-linear material in the case of plane deformation as [7]:

$$W = \lambda_2 S_2 + 1/2\lambda_1 S_1^2 + \lambda_0 S_0 \tag{1.8}$$

where, $S_0 = \vec{c}\cdot\tilde{U}^2\cdot\vec{c}$ is an additional invariant of deformation, due to anisotropy. $\vec{c}$ is the unit vector characterising the direction of anisotropy. $\lambda_0, \lambda_1, \lambda_2$ are the material constants. In the case of randomly unidirectional fibre reinforced composite or a lamina composite the material constants are the effective moduli [8,9]:

$$\lambda_2 = \langle\mu\rangle, \quad \lambda_1 = \langle\lambda\rangle + \frac{\langle\frac{\lambda}{(\lambda+2\mu)}\rangle^2}{\langle\frac{1}{(\lambda+2\mu)}\rangle} - \langle\frac{\lambda^2}{(\lambda+2\mu)}\rangle, \quad \lambda_3 = \frac{1}{\langle\frac{1}{\mu}\rangle}, \quad \lambda_o = \lambda_o(\lambda_2, \lambda_3) \tag{1.9}$$

and we note that in the case of degeneracy into isotropy, the energy function (1.8) automatically reduces to the energy (1.7) and accordingly for the effective moduli $\lambda_3, \lambda_2, \lambda_1$, while $\lambda_o = 2(\lambda_3 - \lambda_2)$ vanishes, i.e.

$$\lambda_3 = \lambda_2 = \mu, \quad \lambda_1 = \lambda, \qquad \lambda_o = 0. \tag{1.9$'$}$$

For any finite function $\varphi(\vec{\xi}, t) \in \Omega \times [0, T)$, $\langle\varphi\rangle$ denotes its geometric average over Ω with the volume $|\Omega| : \langle\varphi\rangle = \frac{1}{|\Omega|}\int_\Omega \varphi \, d\,\Omega$.

Now, invoking the hypothesis of hyperelasticity of Cauchy-Truesdell [10], we take the Frechet derivative [3,11] of the energy with respect to the geometry of deformation (the deformation gradient) $\overset{o}{\nabla}\vec{R}$ and obtain Piola stress tensor $\tilde{P}$, to which it is energy conjugate:

$$\tilde{P} = \frac{\partial W}{\partial \overset{o}{\nabla}\vec{R}} = 2\lambda_2 \overset{o}{\nabla}\vec{R} + (\lambda_1 S_1 - 2\lambda_2)\tilde{O}^D + \lambda_0 \vec{c}\vec{c}\cdot \overset{o}{\nabla}\vec{R}. \tag{1.10}$$

In the absence of body force, we obtain the static equation and the accompanying boundary condition:

$$\overset{o}{\nabla}\tilde{P} = 0$$

$$\vec{f} d\Sigma = \vec{n} \cdot \tilde{P} d\Sigma_o$$

where, $d\Sigma$ is the element of the boundary in the current configuration on which the force $\vec{f}$ acts while $d\Sigma_o$ is an element of the boundary in the reference configuration, with the normal vector $\vec{n} = (n_1, n_2)$.

The component form of these relations are:

$$\frac{\partial P^{11}}{\partial a^1} + \frac{\partial P^{21}}{\partial a^2} = 0; \quad \frac{P^{12}}{\partial a^1} + \frac{\partial P^{22}}{\partial a^3} = 0; \quad \frac{\partial P^{33}}{\partial a^3} = 0 \tag{1.11}$$

and

$$f_1 \frac{d\Sigma}{d\Sigma_o} = n_1 P^{11} + n_2 P^{21}; \quad f_2 \frac{d\Sigma}{d\Sigma_o} = n_1 P^{12} + n_2 P^{22}. \tag{1.11$'$}$$

2 COMPLEX VARIABLE FORMULATION

Complex Variable Representation of Static Equation

We now look at Ω as a subspace of the complex plane $\mathcal{C}$. In place of the material coordinates a^1, a^2 and x^1, x^2 we introduce the complex variables:

$$\zeta = a^1 + ia^2 \in \Omega_o \;\; and \;\; z = x^1 + ix^2 \in \Omega. \tag{2.1}$$

where, $i = \sqrt{-1}$ is the imaginary unit. Then,

$$\frac{\partial}{\partial\zeta} = \frac{1}{2}(\frac{\partial}{\partial a^1} - i\frac{\partial}{\partial a^2}); \quad \frac{\partial}{\partial\overline{\zeta}} = \frac{1}{2}(\frac{\partial}{\partial a^1} + i\frac{\partial}{\partial a^2}) \tag{2.2}$$

and

$$2\frac{\partial z}{\partial\zeta} = (\frac{\partial x^1}{\partial a^1} + \frac{\partial x^2}{\partial a^2}) + i(\frac{\partial x^2}{\partial a^1} - \frac{\partial x^1}{\partial a^2})$$

or, in view of (1.6)

$$\frac{\partial z}{\partial\zeta} = \frac{1}{2}q\exp{(i\chi)} \quad and \quad \exp{(i\chi)} = \frac{\partial z}{\partial\zeta}|\frac{\partial z}{\partial\zeta}|. \tag{2.3}$$

The Piola stress tensor (1.10), the static equations (1.11) and the boundary conditions (1.11)' respectively become:

$$P^{11} + iP^{12} = \phi(q)\exp{(i\chi)} + 2i\lambda_2\frac{\partial z}{\partial a^2} + 2\lambda_o(q\exp{(i\chi)} + i\frac{\partial z}{\partial a^2})$$

$$P^{22} - iP^{21} = \phi(q)\exp{(i\chi)} - 2\lambda_2\frac{\partial z}{\partial a^1} \tag{2.4}$$

$$P^{33} = P^{33}(a^1, a^2);$$

$$\frac{\partial\Phi(\zeta)}{\partial\overline{\zeta}} = -2\lambda_0\frac{\partial^2 z}{\partial a^1\partial a^1} \tag{2.5}$$

and

$$if\frac{dS}{ds_o} - \frac{d\zeta}{ds_o}\Phi(\zeta) + 4\lambda_2\frac{dz}{ds_o} = 2\lambda_0\frac{\partial z}{\partial a^1}in_1 \tag{2.5$'$}$$

where, $n = n_1 + in_2$; dS and ds_o are the arc elements in the current and reference configurations respectively, and

$$\Phi(\zeta) \equiv \phi(q)\exp{(i\chi)}; \quad \phi(q) = (\lambda_1 + 2\lambda_2)(q-2) + 2\lambda_2 + \lambda_1(k-1). \tag{2.6}$$

Equations (2.5) and (2.5)' constitute the static boundary value problem in complex variable formulation: they replace the equivalent equations (1.11) and (1.11)'.

Anisotropic Expansion of State Variables

We note that if Ω were to be an isotropic body, then the right-hand side in (2.5) and (2.5)' would vanish. This implies that λ_0 is a true parameter of anisotropy. So, we dimensionalize it and expand the state variables in this,

$$\beta = \frac{\lambda_0}{\lambda_1 + 2\lambda_2} < 1 \tag{2.7}$$

:

$$z = z_0 + \beta z_1 + \beta^2 z_2 + \beta^3 z_3 + ...$$

$$\Phi = \Phi_0(\zeta) + \beta\Phi_1(\zeta) + \beta^2\Phi_2(\zeta) + \beta^3\Phi_3(\zeta) + ... \tag{2.8}$$

$$\vec{f} = \vec{f}_0 + \beta\vec{f}_1 + \beta^2\vec{f}_2 + \beta^3\vec{f}_3 + ...$$

Putting (2.8) in (2.5) and (2.5)' we obtain the recurrence system for the static equations and the boundary conditions:

$$\sum_{m=0}^{\infty} \beta^m \mathcal{F}_m = 0; \tag{2.9}$$

$$\sum_{m=0}^{\infty} \beta^m \mathcal{P}_m = 0 \tag{2.9$'$}$$

where,

$$\mathcal{F}_m \equiv \frac{\partial\Phi_m}{\partial\bar{\zeta}} + 2(\lambda_1 + 2\lambda_2)\frac{\partial^2 z_{m-1}}{\partial a^1 \partial a^1}; \quad z_m \equiv 0 \quad if\, m < 0, m = 0, 1, 2, ...$$

$$\mathcal{P}_m \equiv i f_m \frac{dS}{ds_o} - \frac{d\zeta}{ds_o}\Phi_m(\zeta) + 4\lambda_2\frac{dz_m}{ds_o} - 2\lambda_0\frac{\partial z_{m-1}}{\partial a^1} i n_1.$$

The first equation in the recurrence system (2.9) is Laplacian while each of the subsequent ones are Poissonian, with the right hand depending recursively on the solution of the previous equation. Thus, this much is the effect of anisotropy on the medium: and has in no way influenced the fact or exposed the issue of finite deformation. The effect of finite deformation is exposed in what follows.

Complex Potential and Nonlinear Fundamental Problems

We deduce the *Fundamental Problems*, analogous to the Kolosov-Muskelishvili 1st and 2nd fundamental problems for infinitesimal elasticity, given in terms of two complex potentials [1,2,12,13].

Let

$$z_0 \equiv v = v_1 + iv_2; \quad \Phi_0 \equiv F(\zeta) \quad and \quad f_0 \equiv h = h_1 + ih_2. \tag{2.10}$$

Then, the first equation in (2.9) is:

$$\frac{\partial F(\zeta)}{\partial\bar{\zeta}} = 0, \tag{2.11}$$

with the corresponding boundary condition from (2.9)'

$$ih\frac{dS}{ds_o} = \frac{d\zeta}{ds_o}F(\zeta) - 4\lambda_2\frac{dv}{ds_o}. \tag{2.11$'$}$$

By the Cauchy-Riemann equations, (2.11) implies that $F(\zeta)$ is an analytic function of the variable ζ, in the finite plane. It can then be generated in a uniformly convergent series of its argument; for any constants α_m,

$$F(\zeta) = \sum_{m=0}^{\infty} \alpha_m \zeta^m.$$

Now, $F(\zeta)$ is related to the material positions $v(\zeta)$ in the current plane in a pertinent manner. In fact, let $k=1$ in (1.4) and by (2.3), (2.6), (2.10) we put

$$F(\zeta)=\phi_0(q_0)\exp{(i\chi_0)};\quad q_0=2|\frac{\partial v}{\partial\zeta}|;\quad \exp{(i\chi_0)}=\frac{\partial v}{\partial\zeta}/|\frac{\partial v}{\partial\zeta}|. \tag{2.12}$$

Then,

$$F(\zeta)=2(\lambda_1+2\lambda_2)\frac{\partial v}{\partial\zeta}-2(\lambda_1+\lambda_2)\frac{F(\zeta)}{|F(\zeta)|}$$

and

$$(\lambda_1+2\lambda_2)\frac{\partial v}{\partial\zeta}=\lambda_2\varphi'^2(\zeta)+(\lambda_1+\lambda_2)\frac{\varphi'(\zeta)}{\overline{\varphi}'(\zeta)} \tag{2.13}$$

where,

$$F(\zeta)\equiv 2\lambda_2\varphi'^2(\zeta). \tag{2.14}$$

Following Muskhelshivili [1,2,3,12], integrating (2.13) and differentiating the result with respect to $\overline{\zeta}$ we obtain

$$(\lambda_1+2\lambda_2)v(\zeta,\overline{\zeta})=\lambda_2\int\varphi'^2(\zeta)d\zeta-(\lambda_1+\lambda_2)[\frac{\varphi(\zeta)}{\overline{\varphi}'(\zeta)}+\overline{\psi}(\zeta)] \tag{2.15}$$

and

$$-\frac{\lambda_1+2\lambda_2}{\lambda_1+\lambda_2}\frac{\partial v}{\partial\overline{\zeta}}=\frac{\varphi(\zeta)\overline{\varphi}''(\zeta)}{\overline{\varphi}'(\zeta)\overline{\varphi}'(\zeta)}-\overline{\psi}'(\zeta). \tag{2.16}$$

$\psi(\zeta)$ is another analytical function that emerges as a result of integration, in the form of $\overline{\psi}(\zeta)$. Thus, in place of $F(\zeta)$ we now have two analytical functions, called *complex potentials*, through which we can define any particle position $v(\zeta,\overline{\zeta})$ on the current plane Ω.

Now, on any arc length s_0 on the contour Γ_o of the reference configuration we have:

$$\frac{dv}{ds_0}=\frac{\partial v}{\partial\zeta}\frac{d\zeta}{ds_0}+\frac{\partial v}{\partial\overline{\zeta}}\frac{d\overline{\zeta}}{ds_0}.$$

Putting (2.15), (2.16) in this, we obtain

$$-i(\lambda_1+2\lambda_2)\frac{dv}{ds_0}=[\lambda_2\varphi'^2(\zeta)+(\lambda_1+\lambda_2)\frac{\varphi'(\zeta)}{\overline{\varphi}'(\zeta)}]n-(\lambda_1+\lambda_2)[\frac{\varphi(\zeta)\overline{\varphi}''(\zeta)}{\overline{\varphi}'(\zeta)\overline{\varphi}'(\zeta)}+\overline{\psi}'(\zeta)]\overline{n}. \tag{2.17}$$

where, $n=n_1+in_2=-i\frac{d\zeta}{ds_o}$; $\overline{n}=n_1-in_2=i\frac{d\overline{\zeta}}{ds_o}$.

This now enables us to obtain boundary conditions pertinent for equilibrum. By (1.11)' we have

$$[\varphi'^2(\zeta)-\frac{\varphi'(\zeta)}{\overline{\varphi}'(\zeta)}]n-[\frac{\varphi(\zeta)\overline{\varphi}''(\zeta)}{\overline{\varphi}'(\zeta)\overline{\varphi}'(\zeta)}+\overline{\psi}'(\zeta)]\overline{n}=\frac{\lambda_1+2\lambda_2}{2\lambda_2(\lambda_1+\lambda_2)}h\frac{dS}{ds_o}. \tag{2.18}$$

We note that on any contour Γ_o of the cross-section of the body, for any function $\theta(\zeta,\overline{\zeta})$:

$$\int_{\Gamma_o}\theta(\zeta,\overline{\zeta})\overline{n}ds_o=\int_{\Gamma_o}(n_1-in_2)\theta(\zeta,\overline{\zeta})ds_o=\int_{\Sigma_o}(\frac{\partial}{\partial a^1}-i\frac{\partial}{\partial a^2})\theta(\zeta,\overline{\zeta})d\Sigma_o=$$

$$2\int_{\Sigma_o}\frac{\partial\theta(\zeta,\overline{\zeta})}{\partial\zeta}d\Sigma_o \quad and \quad \int_{\Gamma_o}\theta(\zeta,\overline{\zeta})n\,ds_o=\int_{\Sigma_o}\frac{\partial\theta(\zeta,\overline{\zeta})}{\partial\overline{\zeta}}d\Sigma_o. \tag{2.19}$$

Then the resultant force vector $H=\int_\Gamma h(x^1,x^2)dS$ will vanish. Likewise, the resultant moment $M=\int_\Gamma(x^1h_2-x^2h_1)dS=\frac{i}{2}\int_\Gamma(v\overline{h}-\overline{v}h)dS$. Infact,

$$\left(\frac{\lambda_1+2\lambda_2}{2\lambda_2(\lambda_1+\lambda_2)}\right)^{-1}\int_\Gamma h\,dS=2\int_\Sigma\Big[\frac{\partial\varphi'^2(\zeta)}{\partial\overline{\zeta}}+\frac{\varphi'(\zeta)\overline{\varphi}''(\zeta)}{\overline{\varphi}'(\zeta)\overline{\varphi}'(\zeta)}-\frac{\varphi'(\zeta)\overline{\varphi}''(\zeta)}{\overline{\varphi}'(\zeta)\overline{\varphi}'(\zeta)}-\frac{\partial\overline{\psi}(\zeta)}{\partial\zeta}\Big]d\Sigma_o=0$$

and

$$M=i\frac{\lambda_2(\lambda_1+\lambda_2)}{\lambda_1+2\lambda_2}\Big\{\int_{\Gamma_o}\{v\overline{n}[\overline{\varphi}'^2(\zeta)-\frac{\overline{\varphi}'(\zeta)}{\varphi'(\zeta)}]-\overline{v}n[\frac{\overline{\varphi}(\zeta)\varphi''(\zeta)}{\varphi'(\zeta)\varphi'(\zeta)}+\psi'(\zeta)]\}ds_o$$

$$-\frac{i}{2}\int_{\Gamma_o}\{\overline{v}n[\frac{\partial}{\partial\zeta}\frac{\overline{\varphi}(\zeta)}{\varphi'(\zeta)}-\psi(\zeta)]-v\overline{n}[\frac{\partial}{\partial\overline{\zeta}}\frac{\varphi(\zeta)}{\overline{\varphi}'(\zeta)}-\overline{\psi}'(\zeta)]\}ds_o\Big\}.$$

In view of (2.13), (2.16), (2.19) this reduces to

$$M=i\lambda_2\int_{\Gamma_o}[v\overline{n}\overline{\varphi}'^2(\zeta)-\overline{v}n\varphi'^2(\zeta)]ds_o=2i\lambda_2\int_\Sigma\frac{\partial v}{\partial\zeta}\overline{\varphi}'^2(\zeta)-\frac{\partial\overline{v}}{\partial\overline{\zeta}}\varphi'^2(\zeta)]d\Sigma_o$$

and by (2.13) and (2.14), M vanishes.

We carry out the following transformation with the aim of presenting (2.15), (2.18) in a more convenient form.

$$\zeta=r\,\exp(i\theta);\frac{\partial\zeta}{\partial\theta}=i\zeta;\frac{\partial\overline{\zeta}}{\partial\theta}=-i\overline{\zeta}.$$

$$\int\varphi'^2(\zeta)d\zeta=J(\zeta);v'(\zeta,\overline{\zeta})=\frac{\partial v}{\partial\theta};h_o(\zeta,\overline{\zeta})=\frac{k}{2\lambda_2}\frac{dS}{ds_o}(h_1+ih_2). \tag{2.20}$$

$$c\equiv\frac{\lambda_2}{\lambda_1+\lambda_2}=1-2\nu_o;k\equiv\frac{\lambda_1+2\lambda_2}{\lambda_1+\lambda_2}=1+c=2(1-\nu_o);\nu_o\equiv\frac{1}{2}\frac{\lambda_1}{\lambda_1+\lambda_2}.$$

Then, (2.18) becomes

$$J(\zeta)-\frac{\varphi(\zeta)}{\overline{\varphi}'(\zeta)}+\overline{\psi}(\zeta)=\frac{ik}{2\lambda_2}H(\zeta,\overline{\zeta}) \tag{2.21}$$

where,

$$[\varphi'^2(\zeta)-\frac{\varphi'(\zeta)}{\overline{\varphi}'(\zeta)}]n-[\frac{\varphi(\zeta)\overline{\varphi}''(\zeta)}{\overline{\varphi}'(\zeta)\overline{\varphi}'(\zeta)}+\overline{\psi}'(\zeta)]\overline{n}=H(\zeta,\overline{\zeta})$$

and

$$cJ(\zeta)+\frac{\varphi(\zeta)}{\overline{\varphi}'(\zeta)}-\overline{\psi}(\zeta)=kv(\zeta,\overline{\zeta}) \tag{2.22}$$

where,

$$c\varphi'^2(\zeta)+\frac{\varphi'(\zeta)}{\overline{\varphi}'(\zeta)}+\frac{\overline{\zeta}}{\zeta}[\frac{\varphi(\zeta)\overline{\varphi}''(\zeta)}{\overline{\varphi}'(\zeta)\overline{\varphi}'(\zeta)}+\overline{\psi}'(\zeta)]=-ik\frac{v'(\zeta,\overline{\zeta})}{\zeta}.$$

Thus, we obtained two nonlinear fundamental problems (2.21) and (2.22) for plane finite deformation. These are analogous to those already known in the literature for the case of linear elasticity [1].

(i) **Fundamental Problem I:** Find the state of elastic equilibrium of a body when the external load applied on the boundary Γ is specified by (2.21).

(ii) **Fundamental Problem II:** find the state of elastic equilibrium for a body whose contour deforms from Γ_o into Γ, given by the position function (2.22).

This implies that the solution of plane problems of finite elasticity impinges on knowing $\varphi(\zeta)$ and $\psi(\zeta)$; which can be established by their values on the appropriate boundary contours, on the basis of the Cauchy integral theorem, due to their analyticity property.

Physical Components of Stress

It is desirable to express the physical components of stress in the current configuration, since, in practice, it is in this form that they are often made use of. For this purpose, we make use of the expression connecting the Cauchy stress $\tilde{T}$ and the Piola stress $\tilde{P}$, noting that $\sqrt{g} \equiv |\frac{\partial x^\alpha}{\partial a^\beta}| = \frac{\partial v}{\partial \zeta}\frac{\partial \overline{v}}{\partial \overline{\zeta}} - \frac{\partial v}{\partial \overline{\zeta}}\frac{\partial \overline{v}}{\partial \zeta}$:

$$\tilde{T} = \frac{1}{\sqrt{g}} \overset{o}{\nabla} \vec{R}^T \cdot \tilde{P} = \frac{1}{\sqrt{g}}(\frac{\partial x^\alpha}{\partial a^\beta} P^{\beta\gamma} \vec{e_\alpha} \vec{e_\gamma} + P^{33} \vec{e_3}\vec{e_3}); \alpha, \beta, \gamma = 1, 2$$

and

$$t_1 + t_2 = \vec{e_1} \cdot \tilde{T} \cdot \vec{e_1} + \vec{e_2} \cdot \tilde{T} \cdot \vec{e_2} = \frac{1}{\sqrt{g}} \frac{\partial x^\alpha}{\partial a^\beta} P^{\alpha\beta},$$

$$t_1 + t_2 + 4\lambda_2 = \frac{1}{\sqrt{g}} q_0 \phi(q_0),$$

$$t_2 - t_1 - 2it_{12} = \vec{e_2} \cdot \tilde{T} \cdot \vec{e_2} - \vec{e_1} \cdot \tilde{T} \cdot \vec{e_1} - i(\vec{e_1} \cdot \tilde{T} \cdot \vec{e_2} + \vec{e_2} \cdot \tilde{T} \cdot \vec{e_1})$$

$$= \frac{1}{\sqrt{g}}[\frac{\partial x^2}{\partial a^\alpha} P^{\alpha 2} - \frac{\partial x^1}{\partial a^\alpha} - i(\frac{\partial x^1}{\partial a^\alpha} P^{\alpha 2} + \frac{\partial x^2}{\partial a^\alpha} P^{\alpha 1})]$$

$$= \frac{1}{\sqrt{g}} \phi(q_0) \exp(i\chi_0)(\frac{\partial v}{\partial a^1} + i\frac{\partial v}{\partial a^2}) = -\frac{4}{\sqrt{g}} \frac{\phi(q_0)}{q_0} \frac{\partial v}{\partial \zeta} \frac{\partial v}{\partial \overline{\zeta}}.$$

On the contour Γ we have,

$$t_N + t_{NS} = 2\lambda_2 \varphi'^2(\zeta) \frac{\partial \overline{v}}{\partial S} \frac{\partial \zeta}{\partial S} - 2\lambda_2. \tag{2.23}$$

3 Some Conditions for Solution

Now, we attempt to give the possible procedure for solutions, having obtained the fundamental problem one **(F P I)** - (2.21) and the fundamental problem two **(F P II)** - (2.22).

Solution of the fundamental problems centres on the appropriate specification of the two functions $\varphi(\zeta)$ and $\psi(\zeta)$. Since these are supposed analytic functions, the first point of call for their specification is to present them as a uniformly convergent series:

$$\varphi(\zeta) = \sum_{m=0}^{\infty} a_m \zeta^m \tag{3.1}$$

$$\psi(\zeta) = \sum_{m=0}^{\infty} b_m \zeta^m \tag{3.2}$$

where, a_m, b_m are constants and m are integers.

These constants are found from physical abstractions of a specific problem vis-a-vis size of the medium, state of equilibrum etc. We can mention two situations with respect to size.

a. Body with finite size. Here, we can identify the following conditions:
i. prior to loading from the natural state we expect the reference and current configurations to coincide,

$$v(\zeta, \overline{\zeta}) = \zeta. \tag{3.3}$$

ii. at equilibrium we expect the resultant force vector to vanish in Ω,

$$H = \int_{\Gamma} h dS = 0. \tag{3.4}$$

Then, (3.1), (3.2) can be expressed as

$$\varphi(\zeta) = \sum_{m=1}^{\infty} a_m \zeta^m \quad a_1 \equiv 1 \tag*{(3.1)'}$$

$$\psi(\zeta) = \sum_{m=1}^{\infty} b_m \zeta^m. \tag*{(3.2)'}$$

b. Body with infinite size. For an infinite sized body, with or without holes/inclusions, we can identify the conditions:
i. while condition (3.4) may not be pertinent in this case, condition (3.3) will hold in both the unloaded state and at infinity,

$$v(\zeta, \overline{\zeta}) = \zeta \quad when \quad |\zeta| \longrightarrow \infty. \tag{3.5}$$

By (3.5) and (2.22) it is obvious that for $\psi(\zeta)$, any analytic function that vanishes at infinity will do; while $\varphi(\zeta)$, by (3.3) and (2.21), can be taken as any meromorphic funtion with pole order one at infinity:

$$\varphi(\zeta) = \zeta + a_0 + \varphi_0(\zeta) \tag{3.6}$$

where, $\varphi_o(\zeta)$ is analytic and vanishes with its derivatives at infinity and in (3.1) $a_1 = 1; a_0$ is any constant.

In fact, putting (3.6) in (2.21), (2.22) and taking cognizance of

$$\varphi'(\zeta) = 1 + \varphi_o'(\zeta); \varphi'^2(\zeta) = 1 + 2\varphi_o'(\zeta) + {\varphi_o'}^2(\zeta);$$

$$J(\zeta) = \zeta + b + 2\varphi_o(\zeta) + J_o(\zeta); J_o(\zeta) = \int \varphi_o'(\zeta)d\zeta; \frac{\varphi(\zeta)}{\overline{\varphi}'(\zeta)} = \frac{\zeta + a + \varphi_o(\zeta)}{1 + \overline{\varphi}_o'(\zeta)}$$

we obtain

$$J_o(\zeta) + 2\varphi_o(\zeta) + \zeta + b - \frac{\varphi_o(\zeta) + \zeta + a}{1 + \overline{\varphi}_o'(\zeta)} + \overline{\psi}(\zeta) = \frac{ik}{2\lambda_2}H(\zeta, \overline{\zeta}), \tag{3.7}$$

$$c[J_o(\zeta) + 2\varphi_o(\zeta) + \zeta + b] + \frac{\varphi_o(\zeta) + \zeta + a}{1 + \overline{\varphi}_o'(\zeta)} - \overline{\psi}(\zeta) = kv(\zeta, \overline{\zeta}). \tag{3.8}$$

By (3.5) we consequently obtain the values for a and b from (3.7), (3.8): $b - a = \frac{ik}{2\lambda_2}H_\infty; (c+1)\zeta + cb + a = k\zeta$ and

$$a = -\frac{ic}{2\lambda_2}H_\infty; b = \frac{i}{2\lambda_2}H_\infty \tag{3.9}$$

where, $H_\infty = {H^1}_\infty + i{H^2}_\infty$ is the resultant force at infinity.

4 General Boundary Condition for Fundamental Problem II

Consider a body whose cross-section Σ_o in the reference configuration is bounded by a circular contour Γ_o. We assume that this bounding contour is isomorphic to a circle of unit radius and let

$$\zeta = \exp(i\theta) \equiv \sigma; \quad n = \sigma, \overline{n} = \sigma^{-1}; \quad |\zeta| = 1.$$

Then, by analyticity of $\varphi(\zeta), \psi(\zeta)$ we use their behaviour and values on the boundary Γ to establish them on Σ as a whole.

Finite Domain

For a finite domain, (3.1)' and (3.2)' are written in the form

$$\varphi(\zeta) = \sum_{m=1}^{\infty} a_m\zeta^m \qquad a_1 \neq 0, \tag{4.1}$$

$$\psi(\zeta) = \sum_{m=1}^{\infty} b_m\zeta^m. \tag{4.2}$$

Then,

$$\varphi'(\zeta) = \sum_{m=0} (1 + m)a_{1+m}\zeta^m, \tag{4.3}$$

$$\varphi'^2(\zeta) = \sum_{m=0} [\sum_{t=0} (1 + t)(1 + m - t)a_{1+t}a_{1+m-t}]\zeta^m = \sum_{m=0} c_m\zeta^m, \tag{4.4}$$

$$c_m \equiv \sum_{t=0}(1+t)(1+m-t)a_(1+t)a_(1+m-t), \tag{4.4$'$}$$

$$J(\zeta) = \sum_{m=1} \frac{1}{m} c_{m-1}\zeta^m. \tag{4.5}$$

Now, on the boundary contour put $\zeta = \sigma$. The conjugate values of expressions take a more convenient form:

$$\overline{\varphi}'(\sigma) = \sum_{m=0}(1+m)\overline{a}_{1+m}\sigma^{-1}; \quad \frac{1}{\overline{\varphi}'(\sigma)} = \sum_{m=0} d_{-m}\sigma^m \tag{4.6}$$

where,

$$d_{-m} = -\frac{1}{\bar{a}_1}\sum_{t=1}\bar{a}_{1+t}d_{-m+t}; \quad d_0 = \frac{1}{\bar{a}_1}. \tag{4.7}$$

From (4.1), (4.6) we can see that,

$$\frac{\varphi(\sigma)}{\overline{\varphi}'(\sigma)} = d_0 \sum_{m=1}^{\infty} a_m\sigma^m + \sum_{\infty}^{\infty} l_m\sigma^m \tag{4.8}$$

where, l_m are the Laurent's series coefficients:

$$l_m = \sum_{t=1}^{\infty} a_{m+t}d_{-t}; \quad l_{-m} = \sum_{t=1}^{\infty} a_t d_{-m-t}. \tag{4.9}$$

Thus, the general boundary conditions for the fundamental problem-two can be written out.

Boundary Condition for Finite Domain. In view of (2.22), we have

$$cJ(\sigma) + \frac{\varphi(\sigma)}{\overline{\varphi}'(\sigma)} - \overline{\psi}(\sigma) = \sum_{-\infty}^{\infty} U_m\sigma^m. \tag{4.10}$$

Putting (4.2), (4.5), (4.8) in (4.10), we obtain

$$c\sum_{-m}^{\infty}\frac{c_{m-1}}{m}\sigma^m + d_0\sum_{m=1}^{\infty} a_m\sigma^m + \sum_{-\infty}^{\infty} l_m\sigma^m - \sum_{m=1}^{\infty}\overline{b}_m\sigma^{-m} = \sum_{-\infty}^{\infty} U_m\sigma^m; \tag{4.11}$$

$$l_0 = U_0; \quad \frac{cc_{m-1}}{m} + \frac{a_m}{\overline{a}_1} + l_m = U_m; \quad l_{-m} - \overline{b}_m = U_{-m}. \tag{4.12}$$

We note that U_0 corresponds to rigid displacement. For this, since we consider finite region, $U_0 = l_0 = 0$.

Infinite Domain with a Circular Hole

Here, in view of (3.1), (3.2), (3.6), (3.9) we have:

$$\varphi_o(\zeta) = \sum_{m=1}^{\infty} a_m\zeta^{-m}, \tag{4.13}$$

$$\psi(\zeta) = \sum_{b_{-m}}^{\infty}\zeta^{-m}, \tag{4.14}$$

$$\varphi_o'(\zeta) = -\frac{1}{\zeta}\sum_{m=1}^{\infty} m a_{-m}\zeta^{-m}; \quad \varphi'(\zeta) = 1 + \varphi_o'(\zeta) = 1 - \frac{1}{\zeta}\sum_{m=1}^{\infty} m a_{-m}\zeta^{-m}. \tag{4.15}$$

$$\varphi'^2_o(\zeta) = \frac{1}{\zeta^4}\sum_{m=o}^{\infty} c_{-m}\zeta^{-m}; \quad J_o(\zeta) = -\frac{1}{\zeta^3}\sum_{m=0}^{\infty}\frac{c_{-m}}{3+m}\zeta^{-m} \tag{4.16}$$

where,

$$c_{-m} = \sum_{t=0}^{m}(1+m)(1+m-t)a_{-1-t}a_{-1-m+t}. \tag{4.17}$$

On the unit radius boundary contour we have:

$$\overline{\varphi}'(\sigma) = 1 - \sigma\sum_{m=1}^{\infty} m\overline{a}_{-m}\sigma^m; \quad \frac{1}{\overline{\varphi}'(\sigma)} = \sum_{m=0}^{\infty} d_m\sigma^m, \tag{4.18}$$

where,

$$d_m = \sum_{t=1}^{m-1} t\overline{a}_{-t}d_{m-t-1}; \quad d_0 = 1, d_1 = 0. \tag{4.19}$$

Consequently,

$$\frac{\varphi(\sigma)}{\overline{\varphi}'(\sigma)} = (\sigma + a + \sum_{m=1}^{\infty} a_{-m}\sigma^{-m})(1 + \sum_{m=1}^{\infty} d_m\sigma^m)$$

$$= (\sigma + a)(1 + \sum_{m=1}^{\infty} d_m\sigma^m) + \sum_{m=1}^{\infty} a_{-m}\sigma^{-m} + \sum_{-\infty}^{\infty} l_m\sigma^m \tag{4.20}$$

where,

$$l_m = \sum_{t=1}^{\infty} a_{-t}d_{m+t}; \quad l_{-m} = \sum_{t=0}^{\infty} a_{-t-m}d_m. \tag{4.21}$$

Again, the general boundary conditions for fundamental problem-two, for infinite region, can be written out.

Boundary Condition for Infinite Domain. By analogy to (4.10), (4.12) we have

$$cJ(\sigma) + \frac{\varphi(\sigma)}{\overline{\varphi}'(\sigma)} - \overline{\psi}(\sigma) = \sum_{-\infty}^{\infty} U_m\sigma^m \tag{4.22}$$

and

$$k\sigma + (1+2c)\sum_{m=1} \infty a_{-m}\sigma^{-m} - (a+\sigma)\sum_{m=1}^{\infty} d_m\sigma^m - c\sum_{m=0}^{\infty}\frac{c_{-m}}{3+m}\sigma^{-3-m}$$

$$-\sum_{m=1} \infty\overline{b}_{-m}\sigma^m + \sum_{-\infty}^{\infty} l_m\sigma^m = \sum_{-\infty}^{\infty} U_m\sigma^m; \tag{4.23}$$

$$l_0 = U_0; \quad k + l_1 - b_{-1} = U_1; \quad (1+2c)a_{-1} + l_{-1} = U_{-1}; \quad (1+2c)a_{-2} + l_{-2} = U_{-2};$$

$$ad_m + d_{m-1} + l_m - \overline{b}_{-m} = U_m, \quad if \quad m \geq 2; \tag{4.24}$$

$$(1+2c)a_{-m} - \frac{c}{3+m}c_{-m} + l_{-m} = U_{-m} \quad if \quad m \geq 3.$$

With the recurrence systems (4.12), and (4.24) we obtain relations which enable us to evaluate coefficients $a_i, b_i \; i = \pm m$ of the series (3.1), (3.2) for fundamental problem-two; hence, the analytical functions $\varphi(\zeta)$ and $\psi(\zeta)$.

5 Solution of Fundamental Problem II for an Infinite Region With a Hole

Consider the equilibrum of an infinite transversely-isotropic plane; in the current configuration having a unit circular hole, centre at the origin. We are interested in the displacement field developed in the medium using the FP II formulation.

Here, we specify the potential $\varphi(\zeta)$ and scout for the second potential $\psi(\zeta)$, intrincically, using the boundary value. Let from (3.6),(4.13)

$$\varphi(\zeta) = \zeta + \frac{a_m}{\zeta^m}; \quad \zeta = r\exp(i\theta), \quad 1 \leq r < \infty, \tag{5.1}$$

a_m are real constants, $m = 1, 2, 3, ...$ Then,

$$\varphi'(\zeta) = 1 - ma_m\zeta^{-1-m} = 1 - a\zeta^{-m-1}, \quad a \equiv ma_m,$$

$$\varphi'^2(\zeta) = 1 - 2a\zeta^{-1-m} + a^2\zeta^{-2(1+m)}; \quad J(\zeta) = \zeta + \frac{2a}{m} - \frac{a^2}{1+2m}\zeta^{-2(1+m)}. \tag{5.2}$$

Specification of Potential by Boundary Value

On the boundary contour, $r = 1$, we have

$$\varphi(\sigma) = \sigma + \frac{a}{m}\sigma^{-m}; \quad \varphi'(\sigma) = 1 - a\sigma_{-(1+m)}; \quad \overline{\varphi}'(\sigma) = 1 - a\sigma^{1+m}; \tag{5.2$'$}$$

$$\frac{\varphi(\sigma)}{\overline{\varphi}'(\sigma)} = (\sigma + \frac{a}{m}\sigma^{-m})(1 - a\sigma^{1+m})^{-1}.$$

Demand that $|a| < 1$ and generate the second factor in the binomial expansion: $(1 - a\sigma^{1+m})^{-1} = \sum_{n=0}(a\sigma^{1+m})^n$. Then,

$$\frac{\varphi(\sigma)}{\overline{\varphi}'(\zeta)} = \sigma + \frac{a}{m}\sigma^{-m} + \frac{a^2}{m}\sigma + (1 + \frac{a^2}{m})\sum_{n=1}^{\infty} a^n\sigma^{(1+m)n+1}. \tag{5.3}$$

The boundary value for the particle position, by (2.22) and in view of the above become

$$kv(\sigma) = (k + \frac{a^2}{m})\sigma + \frac{1+2c}{m}a\sigma^{-m} - \frac{ca^2}{1+2m}\sigma^{-1-2m} + (1 + \frac{a^2}{m})\sum_{n=1} a^n\sigma^{(1+m)n+1} - \overline{\psi}(\sigma). \tag{5.4}$$

This provides the basis for specifying $\psi(\zeta)$ as

$$\psi(\zeta) = \frac{a^2}{m\zeta} + (1 + \frac{a^2}{m})\sum_{n=1} a^n\zeta^{-(1+m)n+1} \tag{5.5}$$

such that on the boundary contour

$$\psi(\sigma) = \frac{a^2}{m}\sigma^{-1} + (1 + \frac{a^2}{m})\sum_{n=1} a^n \sigma^{-(1+m)n+1}; \tag{5.5$'$}$$

and this in turn enables the boundary expression for $v(\zeta, \overline{\zeta})$ to take a simpler form:

$$v(\sigma) = \sigma + b\sigma^{-m} - d\sigma^{-1-2m}; \quad b \equiv \frac{1+2c}{mk}a, \;\; d \equiv \frac{c}{(1+2m)k}a^2. \tag{5.4$'$}$$

Now, the specification in (5.5) also provides a basis to consider the series

$$S = \sum_{n=1}^{\infty} \rho^n \exp(i\phi); \quad 0 \le \rho \le a \quad as \quad 1 \le r < \infty, \;\; a < 1 \tag{5.6}$$

where,

$$\zeta = \rho\exp(i\phi); \;\; \rho \equiv ar^{-1-m}, \; \phi \equiv -(1+m)\theta. \tag{5.7}$$

S has a finite sum. In fact, note that

$$\sum \rho^n \sin(n\phi) = q\rho\sin\phi; \quad \sum \rho^n \cos(n\phi) = q\rho(\cos\phi - \rho);$$

$$q \equiv \frac{1}{1 - 2\rho\cos\phi + \rho^2} = \frac{1}{(\exp(i\phi) - \rho)(\exp(-i\phi) - \rho)}.$$

Then,

$$S = \frac{\rho(\exp(i\phi - \rho))}{\exp(i\phi) - \rho} = \frac{a}{\zeta^{1+m} - a}.$$

This implies that (5.5) can take the simpler form

$$\psi(\zeta) = \frac{a(a\zeta^{1+m} + m)}{m\zeta(\zeta^{1+m} - a)} \tag{5.8}$$

Displacement and Stress Field Having obtained the specification for the potential $\psi(\zeta)$ and thus $v(\zeta, \overline{\zeta})$:

$$kv(\zeta, \overline{\zeta}) = c(\zeta + \frac{2a}{m}\zeta^{-m}) - \frac{ca^2}{(1+2m)}\zeta^{-(1+2m)} + \frac{\overline{\zeta}^{1+m}}{m\zeta^m}\frac{m\zeta^{1+m} + a}{\overline{\zeta}^{1+m} - a} - \frac{a(a\overline{\zeta}^{1+m} + m)}{m\overline{\zeta}(\overline{\zeta}^{1+m} - a)}. \tag{5.9}$$

We can proceed, for the purpose of making subsequent analysis well tractable, to find the pertinent expression for particle position and stress in terms of the transformed radius $\rho = ar^{-1-m}$, given in (5.7). In fact, we note that by (5.2), (5.2)', (5.7)

$$J(\rho, \sigma) = r(\sigma + \frac{2\rho}{m}\sigma^{-m} - \frac{\rho^2}{1+2m}\sigma^{-1-2m}); \;\; \frac{\varphi(\zeta)}{\overline{\varphi'}(\zeta)} = r\frac{\sigma + \frac{1}{m}\rho\sigma^{-m}}{1 - \rho\sigma^{1+m}};$$

$$\overline{\psi}(\zeta) = \frac{\rho}{r}\sigma \times \frac{\frac{\rho}{m}r^{2+2m} + \sigma^{1+m}}{1 - \rho\sigma^{1+m}}. \tag{5.2$''$}$$

Then, on the boundary (5.9) becomes

$$k\frac{v}{r} = c[\sigma + (\frac{2}{m}\sigma^{-m})\rho - \frac{\sigma^{-1-2m}}{1+2m}\rho^2]$$

$$+(1-\rho\sigma^{1+m})^{-1}[(1-r^{-2}\sigma^{1+m}\rho)\sigma + \frac{1}{m}\sigma^{-m}(1-\rho r^{2m}\sigma^{1+m})\rho]. \quad (5.10)$$

But,

$$(1-\rho\sigma^{1+m})^{-1} = (1-\rho\exp(i\phi))^{-1} = q(1-\rho\exp(-i\phi)) = q(1-\rho\sigma^{-(1+m)}).$$

Then, we obtain an exact expression in terms of ρ

$$k\frac{v}{r} = c[\sigma = \rho\frac{2}{m}\sigma^{-m} - \rho^2\frac{\sigma^{-1-2m}}{1+2m}] + q\{[\sigma - \rho\frac{1-m}{m}\sigma^{-m}]$$

$$-\rho\frac{1}{r^2}\sigma^{2+m} + \rho^2\frac{1}{r^2}[(1-\frac{r^{2+2m}}{m})\sigma - \frac{r^2}{m}\sigma^{-1-2m}] + \rho^3\frac{1}{mr^3}r^{3+2m}\sigma^{-m}\}. \quad (5.11)$$

This expression is grossly nonlinear to allow for a tractable analysis. However, using physics of the problem, we can obtain useful information. In fact, note that $r \longrightarrow \infty$ induces $\rho \longrightarrow 0$ and $\frac{1}{r^2} \longrightarrow 0$, while $v \longrightarrow \zeta$; then, terms of $o(\rho^2)$ and $o(\frac{\rho}{r^2})$ are ignored in (5.11):

$$k\frac{v}{r} = (c+q)\sigma + \frac{\rho}{m}[2c-(m-1)q]\sigma^{-m} + o(\rho^2, \frac{\rho}{r^2}). \quad (5.12)$$

On the other hand, as $\rho \longrightarrow a$ we have $r \longrightarrow 1$: i.e. we are on the boundary and $v \longrightarrow v(a) = v(\sigma)$, (5.4)', in the order of $o(\frac{a^2}{r^{2+2m}})$. Then, the previous expression can be written as

$$k\frac{v}{r} = (c+q)\sigma + \frac{\rho}{m}[2c-(m-1)q]\sigma^{-m} + o(\frac{a^2}{r^2}). \quad (5.13)$$

Thus, on the basis of (5.13) we can evaluate the material radius in the current configuration, to obtain the picture of what a unit circle has become due to deformation:

$$R^2 \equiv k^2\frac{v\overline{v}}{r^2} = (c+q)^2 + \frac{\rho^2}{m^2}[2c-(m-1)q]^2 + 2(c+q)\rho[2c-(m-1)q]\cos(1+m)\theta$$

$$= (c+q)^2 + 2(c+q)\rho[2c-(m-1)q]\cos(1+m)\theta + o(\rho^2).$$

So, to the approximation of degree 2, a circle of constant radius in Ω_o maps into a closed curve of degree $1+m$ symmetry in Ω:

$$R = (c+q)^2 + 2(c+q)\rho[2c-(m-1)q]\cos(1+m)\theta. \quad (5.14)$$

Now, on the basis of (2.23), the physical components of the stress are evaluated, differentiating (5.13) and taking cognizance of the following:

$$\frac{\partial v}{\partial\theta} = [i(c+q)+q']\sigma - \{[2c-(m-1)q] - \frac{m-1}{m}q'\}\sigma^{-m},$$

$$-i\frac{k}{r}\frac{\partial v}{\partial\theta} = A\sigma - B\rho\sigma^{-m}; \quad i\frac{k}{r}\frac{\partial v}{\partial\theta} = \overline{A}\overline{\sigma} - \overline{B}\rho\sigma^{m}; \quad q' \equiv \frac{dq}{d\theta}$$

where,

$$A=c+q-iq',\ \overline{A}=c+q+iq',\ B=2c-(m-1)q-i\frac{m-1}{m}q',\ \overline{B}=2c-(m-1)q+i\frac{m-1}{m}q'. \quad (5.15)$$

$$\frac{\partial\zeta}{\partial\theta}=ir\sigma;\quad \frac{k}{r^2}\frac{\partial\overline{v}}{\partial\theta}\frac{\partial\zeta}{\partial\theta}=(\overline{A}\sigma-\overline{B}\rho\sigma^m)\sigma=\overline{A}-\overline{B}\rho\sigma^{1+m}.$$

$$\frac{\partial v}{\partial\theta}\frac{\partial\overline{v}}{\partial\theta}=\frac{r^2}{k^2}(A\sigma-\rho B\sigma^{-m})(\overline{A}\overline{\sigma}-\overline{B}\rho\sigma^m).$$

$$dS^2=dvd\overline{v}=(\frac{\partial v}{\partial r}dr+\frac{\partial v}{\partial\theta}d\theta)(\frac{\partial\overline{v}}{\partial r}dr+\frac{\partial\overline{v}}{\partial\theta}d\theta)=\frac{\partial v}{\partial\theta}\frac{\partial\overline{v}}{\partial\theta}d\theta^2$$

$$=\frac{r^2}{k^2}\{A^2-2\rho[Re\,A\overline{B}\cos(1+m)\theta-iIm\,A\overline{B}\sin(1+m)]\}d\theta^2\equiv R_\theta d\theta^2.$$

$$\frac{\partial\overline{v}}{\partial S}\frac{\partial\zeta}{\partial S}=\frac{1}{R_\theta^2}\frac{r^2}{k}(\overline{A}-\overline{B}\rho\sigma^{1+m})$$

$$=\frac{k(\overline{A}-\overline{B}\rho\sigma^{1+m})}{A^2-2\rho[Re\,A\overline{B}\cos(1+m)\theta-iIm\,A\overline{B}\sin(1+m)\theta]}.$$

Let $t_N\equiv 2\lambda_2\sigma_N$, $t_{NS}\equiv 2\lambda_2\sigma_{NS}$ then,

$$\frac{1}{2\lambda_2}(t_N+it_{NS})=\sigma_N+i\sigma_{NS}$$

$$=\frac{k[\overline{A}-\rho(2\overline{A}\sigma^{-1-m}+B\sigma^{1+m})]}{A^2-2\rho[Re\,A\overline{B}\cos(1+m)\theta-iIm\,A\overline{B}\sin(1+m)\theta]}-1. \quad (5.16)$$

Analysis of Solution

We now consider the analysis of the general solution when $m=1$. For this, all through formulae (5.1) to (5.16) we insert m = 1 and obtain the consequent expressions.

(i.) **Complex Potentials.** From (5.1), (5.8):

$$\varphi(\zeta)=\zeta+\frac{a}{\zeta};\quad \psi(\zeta)=\frac{a(a\zeta^2+1)}{\zeta(\zeta^2-a)}. \quad (5.17)$$

(ii.) **Current Position of Particles.** From (5.13):

$$v(\zeta,\overline{\zeta})=\frac{r}{k}[(c+q+2c\rho)\cos\theta+i(c+q-2c\rho)\sin\theta];\quad q=\frac{1}{1-2\rho\cos 2\theta+\rho^2}. \quad (5.18)$$

(iii.) **Deforming Contour.** From (5.4)'

$$v(\sigma)=(1+b)\cos\theta-d\cos 3\theta+i[(1-b)\sin\theta+d\sin 3\theta];\quad b\equiv\frac{1+2c}{k}a,\ d\equiv\frac{c}{3k}a^2. \quad (5.19)$$

(iv.) **Dimensionless Physical Stress.** From (5.15), (5.16):

$$\sigma_N + i\sigma_{NS} = \frac{k[\overline{A} - \rho(2\overline{A}\sigma^{-2} + B\sigma^2)}{A^2 - 4\rho c\, Re\, A\cos 2\theta} - 1;$$

$$A = c + q - iq', \;\; B = 2c, \;\; q = \frac{1}{1 - 2\rho\cos 2\theta + \rho^2}, \;\; q' = -4\rho q^2 \sin 2\theta. \qquad (5.20)$$

One could observe that in (5.18) a circle $r = const.$ deforms into an ellipse with semi-major and semi-minor axes:

$$x_1(r,0) = \frac{r}{k}(c + q + 2c\rho); \;\; x_2(r, 1/2\pi) = \frac{r}{k}(c + q - 2c\rho). \qquad (5.21)$$

The contour curve in (5.19) is also exactly an ellipse with intercepts $1+b$ and $1-b$, if $d \equiv 0$. Otherwise, it is an ellipse-type curve with intercepts, $1+b-d$, $1-b-d$.

We invoke certain physical properties of curves to obtain deeper analysis, vis-a-vis pertinent constraint on the material constants k, c via the deduced constants b, d, which eventually reveals a as the *parameter of finite deformation.* In fact;

(i.) If we allow equality of intercepts on the axes, we have $b = 0$ and $a = 0$ or $c = -1/2$. That is, by $(2.20)_3$ $\nu = 3/4$. But for real materials, the Poisson's coefficient is $0 < \nu < 1/2$ (or theoretically, $-1 < \nu < 1/2$), though by small deformation theory approximation. So, $a = 0$ corresponds to the case when the curve is a circle; and this is the situation which the small deformation theory gives. But here, it takes other values (i.e. not $a \equiv 0$); and this provides the basis for declaring a as the parameter of finite deformation (or parameter of nonlinearity), for this problem.

(ii.) We demand no self-intersection from the curve. This implies that the minor axis is non-negative $1 - b - d \geq 0$ i.e.

$$a^2 + \frac{3}{c}(1 + 2c)a - \frac{3k}{c} \leq 0 \qquad (5.22)$$

This imposes necessary constraint on the parameter of finite deformation a :

$$a < a_l; \quad a_l = \frac{3}{2}\frac{1 + 2c}{c}\{\sqrt{1 + \frac{4}{3}\frac{ck}{(1 + 2c)^2}} - 1\}.$$

At first approximation, $a_l = \frac{k}{1+2c}$. This has the same effect as either ignoring d in (5.19) or dropping a^2 in (5.22): the deforming contour is elliptic. In figure 2 the ellipses are obtained with variation of the deformation parameter; as $a \longrightarrow 0$, an ellipse turns into a circle.

Conclusion

The obtained result shows the need and how to countenance finite deformation, and thus non-linear effect, in the consideration of problems, particularly in areas of application where high precision is required.

However, for future consideration, there is the equal need for numerical computation at the higher level of iteration to investigate any further effects, especially those which may be due to anisotropy.

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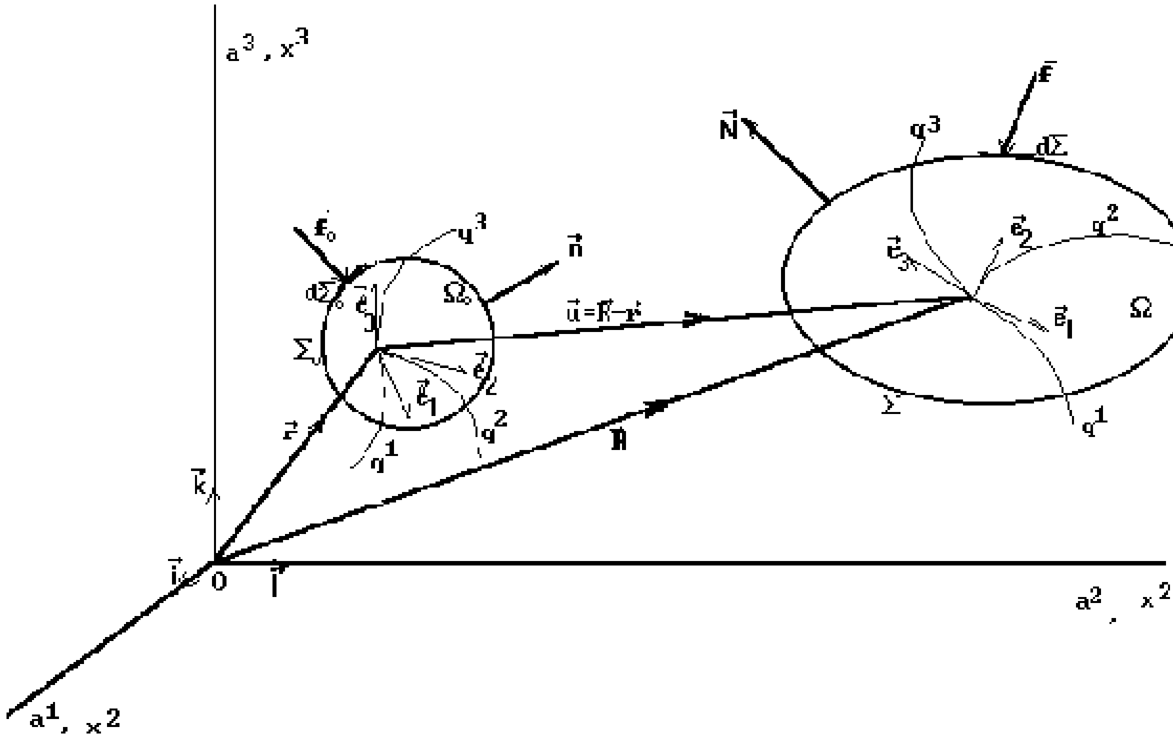
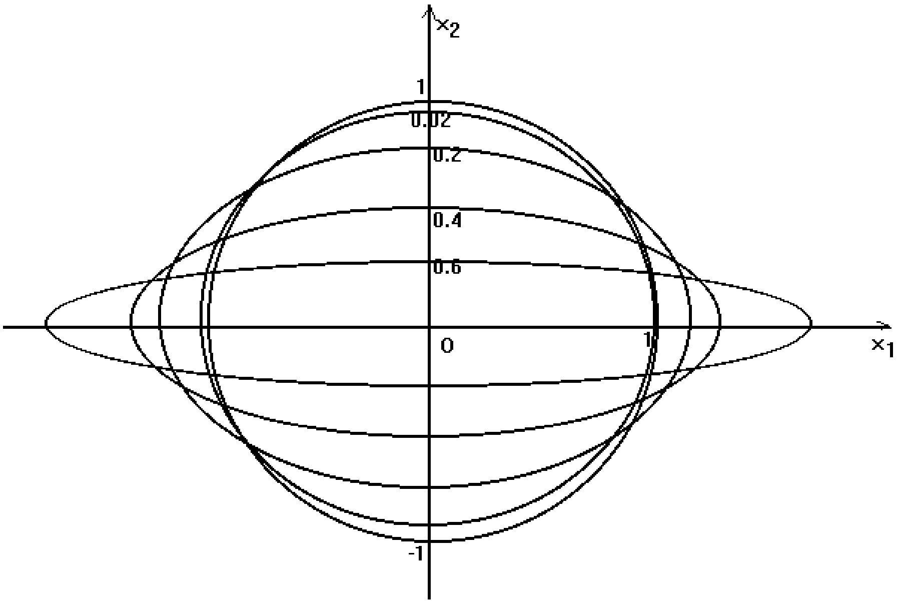
FIGURE CAPTIONS

Figure 1 gives a schematic diagram of the particle positions $\vec{r}$ and $\vec{R}$ respectively in the reference Ω_o and current Ω configurations of a deformable medium. q^m are the material coordinates with the basis $\overset{o}{\vec{e_m}}= \frac{\partial \vec{r}}{\partial q^m} = \vec{i_n} \frac{\partial a^n}{\partial q^m}$ in Ω_o and $\vec{e_m} = \frac{\partial \vec{R}}{\partial q^m} = \vec{i_n} \frac{\partial x^n}{\partial q^m}$ in Ω. a^n, x^n are the rectangular coordinates, with the orthonormal basis $\vec{i_n} = \vec{i}, \vec{j}, \vec{k}; n = 1, 2, 3$. $\vec{u}$ is the displacement vector.

Figure 2 shows the variation of the deforming contour as the parameter a varies; as a increases from zero a circular contour turns into an elliptic one.



a^3, x^3
$\vec{N}$
$\vec{f}$
q^3
$d\Sigma$
$\vec{f}_0$
q^3
$\vec{n}$
$d\Sigma_0$
Ω_0
Σ_0
$\vec{u}=\vec{R}-\vec{r}$
$\vec{e}_2$
q^2
$\vec{e}_3$
$\vec{e}_1$
Ω
$\vec{e}_1$
Σ
q^1
$\vec{r}$
q^2
q^1
$\vec{R}$
$\vec{k}$
$\vec{i}$
0
$\vec{j}$
a^2, x^2
a^1, x^2



x_2
1
0.02
0.2
0.4
0.6
0
1
x_1
-1